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1981 J. Phys. A: Math. Gen. 14 809

(http://iopscience.iop.org/0305-4470/14/4/011)

J. Phys. A: Math. Gen. **14** (1981) 809–819. Printed in Great Britain

Remarks on Turschner's eigenvalue formula

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Received 29 August 1979, in final form 15 October 1980

Abstract. We compare Turschner's approximation to the eigenvalues of Hamiltonians $-\nabla^2+c|x|^\nu$ with previously published numerical results. We examine evidence that Turschner's approximation yields a lower bound to the ground state energy both for boson and fermion systems. We discuss the convergence of Turschner's approximation to the exact energies $E_n^{(\nu)}$ as $n\to\infty$. We also derive a simple expression for Turschner's approximate eigenvalues in terms of the hypergeometric function.

Recently an article appeared in this journal that presented a method for calculating the eigenvalues of Hamiltonians

$$-\nabla^2+c|x|^\nu \tag{1}$$

in $\mathbb{R}$ for $c>0$, $\nu>0$ (Turschner 1979). Soon after its publication it was realised by Crowley and Hill (1979) and also by us that Turschner's method does not yield exact eigenvalues. Crowley and Hill have pointed out a flaw in Turschner's derivation.

Nonetheless, we emphasise with Crowley and Hill that Turschner's method seems to yield excellent approximations to the eigenvalues of Hamiltonians of the above form (1). Crowley and Hill have demonstrated that Turschner's method is in some sense a semiclassical approximation.

We would like to present numerical evidence that the 'eigenvalues' $\tilde{E}_n^{(\nu)}$ obtained by Turschner (equation 4.14) converge to the exact eigenvalues $E_n^{(\nu)}$ as n becomes infinite for a range of ν including 1, 2, and 4. We note that $\tilde{E}_0^{(\nu)}$ seems to be a lower bound to $E_0^{(\nu)}$ for all ν; this result would imply that the Turschner approximation to the ground state energy of boson systems would be a lower bound to the exact ground state energy. It also seems that for $\nu\geqslant 1$,

$$\sum_{n=0}^{N-1}\tilde{E}_n^{(\nu)}\leqslant\sum_{n=0}^{N-1}E^{(\nu)} \tag{2}$$

for all $N\geqslant 1$, i.e. that the ground state of an N-fermion system in the Turschner approximation is a lower bound to the exact ground state energy. Given the well known difficulty of obtaining useful lower bounds to energies of many-particle systems, the discovery of a simple method for constructing lower bounds is of interest.

We have also obtained a particularly simple representation of Turschner's approximate energies in terms of the hypergeometric function. Turschner's approximate

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0305-4470/81/040809+11$01.50

eigenvalues $\tilde{E}_n^{(\nu)}$ of $p^2+|x|^\nu$ are given by

$$\tilde{E}_n^{(\nu)}=2(4b_\nu)^{-2\nu/(\nu+2)}\Gamma\left(1+\frac{2\nu}{\nu+2}\right)\frac{(-1)^n}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{(2+s)^n}{s^{1+2\nu/(\nu+2)}}\right]_{s=2}$$

where

$$b_\nu=\frac{1}{2\pi}\int_0^1\mathrm{d}q\,(1-q^\nu)^{1/2}=\frac{1}{2\pi}\frac{1}{\nu}B\left(\frac{1}{\nu},\frac{3}{2}\right)=\frac{1}{2\pi\nu}\frac{\Gamma(1/\nu)\Gamma(\frac{3}{2})}{\Gamma(1/\nu+\frac{3}{2})}$$

(Turschner 1979, equation 4.14). By Leibnitz' rule,

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}s_n}\frac{(2+s)^n}{s^{1+2\nu/(\nu+2)}}&=\sum_{j=0}^{n}\binom{n}{j}\left(\frac{\mathrm{d}^j}{\mathrm{d}s^j}s^{-[1+2\nu/(\nu+2)]}\right)\left(\frac{\mathrm{d}^{n-j}}{\mathrm{d}s^{n-j}}(2+s)^n\right)\\&=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(1+\frac{2\nu}{\nu+2}\right)_j s^{-[1+2\nu/(\nu+2)]-j}\frac{n!}{j!}(2+s)^j\end{aligned}$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$ for $a>0$, $(a)_k=(-1)^k\Gamma(1-a)/\Gamma(1-a-k)$ for $a\leqslant 0$. Since $(-1)^j n!/(n-j)!=(-n)_j$, it follows that

$$\frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{(2+s)^n}{s^{[1+2\nu/(\nu+2)]}}\right]_{s=2}=2^{-[1+2\nu/(\nu+2)]}\sum_{j=0}^{n}\frac{(-n)_j[1+2\nu/(\nu+2)]_j2^j}{(j!)^2}$$

which we immediately recognise to be $2^{-[1+2\nu/(\nu+2)]}F(-n,1+2\nu/(\nu+2);1;2)$, where F is the hypergeometric function (Gradshteyn and Ryzhik 1965, equation 9.100). Thus after some straightforward algebra, we obtain

$$\begin{aligned}\tilde{E}_n^{(\nu)}=&\left(\frac{\pi}{4}\frac{\Gamma(1/\nu+\frac{3}{2})}{\Gamma(1/\nu+1)\Gamma(\frac{3}{2})}\right)^{2\nu/(\nu+2)}\Gamma(1+2\nu/(\nu+2))\\&\times(-1)^nF(-n,1+2\nu/(\nu+2);1;2).\end{aligned}$$

Since $\tilde{E}_n^{(\nu)}$ is proportional to $(-1)^nF(-n,1+2\nu/(\nu+2);1;2)$ and $F(\alpha,\beta;\gamma;z)$ obeys the Gauss recursion formula

$$\begin{aligned}0=&(\gamma-\alpha)F(\alpha-1,\beta;\gamma;z)\\&+(2\alpha-\gamma-\alpha z+\beta z)F(\alpha,\beta;\gamma;z)+\alpha(z-1)F(\alpha+1,\beta;\gamma;z)\end{aligned}$$

(Gradshteyn and Ryzhik 1965, equation 9.137.2), setting $\alpha=-(n+1)$, $\beta=[1+2\nu/(\nu+2)]$, $\gamma=1$, and $z=2$ implies that

$$0=(n+2)\tilde{E}_{n+2}^{(\nu)}-[1+4\nu/(\nu+2)]\tilde{E}_{n+1}^{(\nu)}-(n+1)\tilde{E}_n^{(\nu)}.$$

Our computation of the values $\tilde{E}_n^{(\nu)}$ given by Turschner's formula were made in double precision (16 significant digits) using this linear recurrence relation.

In table 1 we present evidence that $\tilde{E}_0^{(\nu)}\leqslant E_0^{(\nu)}$ for all positive ν. The exact energies are taken from Barnes *et al* (1976, p 88).

To examine the evidence for inequality (2), we first observe that for $\nu=\infty$, $\tilde{E}_n^{(\infty)}=(\pi^2/4)(n^2+n+\frac{1}{2})$, while $E_n^{(\infty)}=(\pi^2/4)(n^2+2n+1)$. Clearly each $\tilde{E}_n^{(\infty)}$ is a lower bound to $E_n^{(\infty)}$.

For $\nu=4$, in table 3 we compare $\tilde{E}_n^{(4)}$ with $E_n^{(4)}$ and their sums. Although $\tilde{E}_2^{(4)}$ is *not* a lower bound to $E_2^{(4)}$, it is readily seen that inequality (2) holds. The exact energies are from Banerjee *et al* (1978).

Table 1. Eigenvalues of $-d^2/dx^2+|x|^\nu$.

ν	$\tilde{E}_0^{(\nu)}$	$E_0^{(\nu)}$	$E_0^{(\nu)}-\tilde{E}_0^{(\nu)}$	$\tilde{E}_1^{(\nu)}$	$E_1^{(\nu)}$	$E_1^{(\nu)}-\tilde{E}_1^{(\nu)}$	$(E_0^{(\nu)}-\tilde{E}_0^{(\nu)})$ $+(E_1^{(\nu)}-\tilde{E}_1^{(\nu)})$
0.25	1.056 491	1.079 542	0.023 051	1.526 042	1.492 684	−0.033 358	−0.010 306
0.5	1.035 832	1.059 617	0.023 785	1.864 497	1.833 394	−0.031 103	−0.007 318
1	1.006 976	1.018 793	0.011 817	2.349 612	2.338 107	−0.011 505	0.000 312
1.5	0.998 382	1.001 184	0.002 802	2.709 894	2.708 092	−0.001 802	0.001 001
2	1.000 000	1.000 000	0.000 000	3.000 000	3.000 000	0.000 000	0.000 000
3	1.014 530	1.022 948	0.008 418	3.449 401	3.450 563	0.001 162	0.009 580
4	1.032 457	1.060 362	0.027 905	3.785 677	3.799 673	0.013 996	0.041 901
5	1.049 456	1.102 298	0.052 842	4.047 901	4.089 159	0.041 258	0.094 101
6	1.064 576	1.144 802	0.080 226	4.258 303	4.338 599	0.080 296	0.160 523
10	1.108 131	1.298 844	0.190 713	4.801 901	5.097 876	0.295 975	0.486 688
20	1.158 450	1.560 508	0.402 058	5.370 995	6.219 360	0.848 365	1.250 423
50	1.199 772	1.903 191	0.703 419	5.814 282	7.610 400	1.796 118	2.499 537
∞	1.233 701	2.467 401	1.233 701	6.168 503	9.869 604	3.701 102	4.934 802

Table 2. Eigenvalues of $-d^2/dx^2+|x|$.

n	$\tilde{E}_n^{(1)}$	$E_n^{(1)}$	$E_n^{(1)}-\tilde{E}_n^{(1)}$	$\Sigma_{i=0}^{n}(E_i^{(1)}$ $-\tilde{E}_i^{(1)})$	E_n^{WKB}	$E_n^{(1)}-E_n^{\mathrm{WKB}}$
0	1.006 976	1.018 793	0.011 816	0.011 816	1.115 460	−0.096 667
1	2.349 612	2.338 107	−0.011 504	0.000 312	2.320 251	0.017 857
2	3.244 702	3.248 198	0.003 496	0.003 808	3.261 626	−0.013 428
3	4.090 065	4.087 949	−0.002 116	0.001 692	4.081 810	0.006 139
4	4.819 398	4.820 099	0.000 701	0.002 394	4.826 316	−0.006 217
5	5.521 104	5.520 560	−0.000 544	0.001 849	5.517 164	0.003 396
6	6.163 261	6.163 307	0.000 047	0.001 896	6.167 128	−0.003 821
7	6.786 795	6.786 708	−0.000 087	0.001 809	6.784 454	0.002 254
8	7.372 335	7.372 177	−0.000 158	0.001 651	7.374 853	−0.002 676
9	7.944 053	7.944 134	0.000 080	0.001 731	7.942 487	0.001 647
10	8.488 714	8.488 487	−0.000 227	0.001 504	8.490 507	−0.002 020
11	9.022 503	9.022 651	0.000 148	0.001 652	9.021 373	0.001 278
12	9.535 697	9.535 449	−0.000 248	0.001 404	9.537 051	−0.001 602
13	10.039 999	10.040 174	0.000 175	0.001 579	10.039 142	0.001 032
14	10.527 909	10.527 660	−0.000 248	0.001 330	10.528 975	−0.001 315
15	11.008 341	11.008 524	0.000 183	0.001 514	11.007 665	0.000 859
16	11.475 298	11.475 057	−0.000 241	0.001 273	11.476 163	−0.001 107
17	11.935 832	11.936 016	0.000 183	0.001 456	11.935 285	0.000 731
18	12.385 019	12.384 788	−0.000 230	0.001 226	12.385 739	−0.000 950
19	12.828 598	12.828 777	0.000 179	0.001 405	12.828 144	0.000 633
20	13.262 437	13.262 219	−0.000 218	0.001 187	13.263 048	−0.000 829
21	13.691 316	13.691 489	0.000 173	0.001 359	13.690 934	0.000 555
22	14.111 709	14.111 502	−0.000 207	0.001 153	14.112 234	−0.000 732
23	14.527 664	14.527 830	0.000 166	0.001 318	14.527 337	0.000 493
24	14.936 133	14.935 937	−0.000 195	0.001 123	14.936 591	−0.000 653
25	15.340 597	15.340 755	0.000 158	0.001 281	15.340 313	0.000 442
26	15.738 386	15.738 201	−0.000 185	0.001 097	15.738 790	−0.000 589
27	16.132 534	16.132 685	0.000 151	0.001 248	16.132 285	0.000 400
28	16.520 679	16.520 504	−0.000 175	0.001 073	16.521 038	−0.000 534
29	16.905 490	16.905 634	0.000 144	0.001 217	16.905 270	0.000 364
30	17.284 861	17.284 695	−0.000 166	0.001 051	17.285 183	−0.000 488

Table 2—continued.

n	$\tilde{E}_n^{(1)}$	$E_n^{(1)}$	$E_n^{(1)}-\tilde{E}_n^{(1)}$	$\Sigma_{i=0}^{n}(E_i^{(1)}-\tilde{E}_i^{(1)})$	E_n^{WKB}	$E_n^{(1)}-E_n^{WKB}$
31	17.661 162	17.661 300	0.000 138	0.001 189	17.660 966	0.000 334
32	18.032 502	18.032 345	−0.000 157	0.001 032	18.032 793	−0.000 448
33	18.401 001	18.401 133	0.000 132	0.001 163	18.400 825	0.000 308
34	18.764 948	18.764 798	−0.000 150	0.001 014	18.765 213	−0.000 414
35	19.126 255	19.126 380	0.000 126	0.001 139	19.126 096	0.000 285
36	19.483 364	19.483 222	−0.000 143	0.000 997	19.483 606	−0.000 384
37	19.838 009	19.838 130	0.000 120	0.001 117	19.837 865	0.000 265
38	20.188 768	20.188 632	−0.000 136	0.000 981	20.188 989	−0.000 358
39	20.537 218	20.537 333	0.000 115	0.001 096	20.537 086	0.000 247
40	20.882 053	20.881 923	−0.000 130	0.000 967	20.882 257	−0.000 334
41	21.224 719	21.224 830	0.000 111	0.001 077	21.224 599	0.000 231
42	21.564 012	21.563 888	−0.000 124	0.000 953	21.564 201	−0.000 314
43	21.901 261	21.901 368	0.000 106	0.001 059	21.901 150	0.000 217
44	22.235 351	22.235 232	−0.000 119	0.000 940	22.235 527	−0.000 295
45	22.567 511	22.567 613	0.000 102	0.001 042	22.567 408	0.000 205
46	22.896 703	22.896 589	−0.000 114	0.000 928	22.896 867	−0.000 278
47	23.224 067	23.224 165	0.000 098	0.001 026	23.223 972	0.000 193
48	23.548 636	23.548 526	−0.000 110	0.000 916	23.548 789	−0.000 263
49	23.871 470	23.871 564	0.000 095	0.001 011	23.871 382	0.000 183
50	24.191 665	24.191 560	−0.000 106	0.000 905	24.191 809	−0.000 249
51	24.510 210	24.510 301	0.000 091	0.000 997	24.510 128	0.000 173
52	24.826 258	24.826 156	−0.000 102	0.000 895	24.826 393	−0.000 237
53	25.140 733	25.140 821	0.000 088	0.000 983	25.140 656	0.000 165
54	25.452 840	25.452 743	−0.000 098	0.000 885	25.452 968	−0.000 225
55	25.763 446	25.763 531	0.000 085	0.000 970	25.763 374	0.000 157
56	26.071 802	26.071 708	−0.000 094	0.000 876	26.071 922	−0.000 215
57	26.378 723	26.378 805	0.000 082	0.000 958	26.378 655	0.000 150
58	26.683 502	26.683 410	−0.000 091	0.000 867	26.683 615	−0.000 205
59	26.986 906	26.986 985	0.000 080	0.000 946	26.986 842	0.000 143
60	27.288 267	27.288 179	−0.000 088	0.000 858	27.288 375	−0.000 196
61	27.588 311	27.588 388	0.000 077	0.000 935	27.588 251	0.000 137
62	27.886 404	27.886 318	−0.000 085	0.000 850	27.886 506	−0.000 188
63	28.183 231	28.183 306	0.000 075	0.000 924	28.183 174	0.000 131
64	28.478 192	28 478 110	−0.000 083	0.000 842	28.478 289	−0.000 180
65	28.771 937	28.772 009	0.000 072	0.000 914	28.771 883	0.000 126
66	29.063 894	29.063 814	−0.000 080	0.000 834	29.063 987	−0.000 173
67	29.354 680	29.354 751	0.000 070	0.000 904	29.354 630	0.000 121
68	29.643 752	29.643 675	−0.000 078	0.000 827	29.643 841	−0.000 166
69	29.931 696	29.931 764	0.000 068	0.000 895	29.931 648	0.000 116
70	30.217 993	30.217 918	−0.000 075	0.000 819	30.218 078	−0.000 160
71	30.503 202	30.503 269	0.000 066	0.000 886	30.503 157	0.000 112
72	30.786 829	30.786 756	−0.000 073	0.000 812	30.786 910	−0.000 154
73	31.069 404	31.069 469	0.000 064	0.000 877	31.069 361	0.000 108
74	31.350 456	31.350 385	−0.000 071	0.000 806	31.350 534	−0.000 148
75	31.630 493	31.630 556	0.000 063	0.000 869	31.630 452	0.000 104
76	31.909 062	31.908 993	−0.000 069	0.000 799	31.909 136	−0.000 143
77	32.186 649	32.186 710	0.000 061	0.000 860	32.186 609	0.000 101
78	32.462 820	32.462 753	−0.000 067	0.000 793	32.462 891	−0.000 138
79	32.738 040	32.738 100	0.000 059	0.000 853	32.738 002	0.000 097
80	33.011 894	33.011 829	−0.000 066	0.000 787	33.011 963	−0.000 134
81	33.284 827	33.284 885	0.000 058	0.000 845	33.284 791	0.000 094
82	33.556 439	33.556 376	−0.000 064	0.000 781	33.556 505	−0.000 130

Table 2—continued.

n	$\tilde{E}_n^{(1)}$	$E_n^{(1)}$	$E_n^{(1)}-\tilde{E}_n^{(1)}$	$\Sigma_{i=0}^{n}(E_i^{(1)}-\tilde{E}_i^{(1)})$	E_n^{WKB}	$E_n^{(1)}-E_n^{\mathrm{WKB}}$
83	33.827 158	33.827 215	0.000 057	0.000 838	33.827 124	0.000 091
84	34.096 601	34.096 539	−0.000 062	0.000 776	34.096 665	−0.000 125
85	34.365 177	34.365 232	0.000 055	0.000 831	34.365 144	0.000 088
86	34.632 518	34.632 457	−0.000 061	0.000 770	34.632 579	−0.000 122
87	34.899 016	34.899 070	0.000 054	0.000 824	34.898 985	0.000 086
88	35.164 319	35.164 260	−0.000 059	0.000 765	35.164 378	−0.000 118
89	35.428 804	35.428 856	0.000 053	0.000 817	35.428 773	0.000 083
90	35.692 129	35.692 071	−0.000 058	0.000 760	35.692 186	−0.000 114
91	35.954 659	35.954 710	0.000 051	0.000 811	35.954 630	0.000 081
92	36.216 065	36.216 008	−0.000 056	0.000 755	36.216 119	−0.000 111
93	36.476 696	36.476 747	0.000 050	0.000 805	36.476 668	0.000 078
94	36.736 237	36.736 182	−0.000 055	0.000 750	36.736 290	−0.000 108
95	36.995 025	36.995 074	0.000 049	0.000 799	36.994 998	0.000 076
96	37.252 753	37.252 699	−0.000 054	0.000 745	37.252 804	−0.000 105
97	37.509 747	37.509 795	0.000 048	0.000 793	37.509 721	0.000 074
98	37.765 712	37.765 659	−0.000 053	0.000 740	37.765 761	−0.000 102
99	38.020 962	38.021 009	0.000 047	0.000 787	38.020 937	0.000 072

For $\nu = 2$, $\tilde{E}_n^{(2)} = E_n^{(2)}$, so the lower bound is trivial.

For $\nu = 1$, table 2 shows that (2) holds. The exact energies, which are the negatives of the zeros of the Airy function $\mathrm{Ai}(x)$ and its derivative, are from the Royal Society Mathematical Tables (1960).

From table 1 it can be seen that (2) fails for $\nu = \frac{1}{4}$ and $\nu = \frac{1}{2}$ for $N = 2$.

On the basis of the data in tables 1, 2 and 3 and the above discussion, we would like to make the following speculations on the behaviour of the $\tilde{E}_n^{(\nu)}$'s relative to the $E_n^{(\nu)}$'s:

(i) For some critical $\nu_c > 4$, $\tilde{E}_n^{(\nu)} \leqslant E_n^{(\nu)}$ for all $\nu \geqslant \nu_c$ and all n.

(ii) For $2 \leqslant \nu \leqslant \nu_c$, $\tilde{E}_n^{(\nu)} \leqslant E_n^{(\nu)}$ for all n sufficiently large, and (2) obtains for all N.

(iii) For some critical $\nu_c' < 1$, (2) holds for all $\nu_c' \leqslant \nu \leqslant 2$. (2) is likely to be false for all $0 < \nu < \nu_c'$.

The apparent convergence as $n \to \infty$ of $\tilde{E}_n^{(\nu)}$ to $E_n^{(\nu)}$ for $\nu = 1$ and $\nu = 4$ is readily seen from tables 2 and 3. (Convergence here is understood in the sense that the sequence of differences $E_n^{(\nu)} - \tilde{E}_n^{(\nu)}$ goes to zero as n goes to infinity. Note that this is stronger than requiring that the sequence of ratios $E_n^{(\nu)}/\tilde{E}_n^{(\nu)}$ tends to one.) We also see that for $\nu = 1$ and $\nu = 4$, Turschner's approximation yields rather better answers than the WKB approximation. This behaviour can be understood by examining the asymptotic series for $E_n^{(\nu)}$, $\tilde{E}_n^{(\nu)}$, and the WKB energies. We shall analyse the cases $\nu = 1$ and $\nu = 4$ separately. The derivation of these results is presented in the Appendix.

For $\nu = 1$, we have the following asymptotic expansions for the exact energies:

$$E_n^{(1)} = [\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{2/3}\{1 - \tfrac{7}{48}[\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{-2} + \mathrm{O}[(n+\tfrac{1}{2})^{-4}]\}, \qquad n \text{ even}$$

$$E_n^{(1)} = [\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{2/3}\{1 + \tfrac{5}{48}[\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{-2} + \mathrm{O}[(n+\tfrac{1}{2})^{-4}]\}, \qquad n \text{ odd.}$$

The common leading term is the WKB energy

$$E_{n,\mathrm{WKB}}^{(1)} = (\tfrac{3}{4}\pi(n+\tfrac{1}{2}))^{2/3}.$$

Table 3. Eigenvalues of $-d^2/dx^2+x^4$.

n	$\tilde{E}_n^{(4)}$	$E_n^{(4)}$	$E_n^{(4)}-\tilde{E}_n^{(4)}$	$\sum_{i=0}^n (E_i^{(4)} - \tilde{E}_i^{(4)})$	E_n^{WKB}	$E_n^{(4)}-E_n^{WKB}$
0	1.032 457	1.060 362	0.027 905	0.027 905	0.867 145	0.193 217
1	3.785 677	3.799 673	0.013 996	0.041 901	3.751 920	0.047 753
2	7.456 636	7.455 698	−0.000 938	0.040 962	7.413 988	0.041 710
3	11.637 451	11.644 746	0.007 294	0.048 257	11.611 525	0.033 220
4	16.260 141	16.261 826	0.001 685	0.049 942	16.233 615	0.028 211
5	21.234 064	21.238 373	0.004 308	0.054 250	21.213 653	0.024 720
6	26.526 490	26.528 471	0.001 981	0.056 231	26.506 336	0.022 136
7	32.095 455	32.098 598	0.003 143	0.059 374	32.078 464	0.020 134
8	37.921 096	37.923 001	0.001 905	0.061 279	37.904 472	0.018 529
9	43.978 629	43.981 158	0.002 530	0.063 808	43.963 948	0.017 210
10	50.254 483	50.256 255	0.001 771	0.065 580	50.240 152	0.016 102
11	56.732 066	56.734 214	0.002 148	0.067 728	56.719 057	0.015 157
12	63.401 408	63.403 047	0.001 639	0.069 367	63.388 708	0.014 339
13	70.250 509	70.252 395	0.001 885	0.071 253	70.238 771	0.013 623
14	77.271 679	77.273 200	0.001 522	0.072 775	77.260 210	0.012 990
15	84.455 774	84.457 466	0.001 692	0.074 466	84.445 040	0.012 426
16	91.796 647	91.798 067	0.001 420	0.075 886	91.786 147	0.011 919
17	99.287 064	99.288 607	0.001 542	0.077 428	99.277 145	0.011 461
18	106.921 976	106.923 307	0.001 331	0.078 760	106.912 262	0.011 045
19	114.695 495	114.696 917	0.001 422	0.080 182	114.686 253	0.010 664
20	122.603 385	122.604 639	0.001 254	0.081 437	122.594 324	0.010 315
21	130.640 745	130.642 069	0.001 324	0.082 760	130.632 076	0.009 993
22	138.803 961	138 805 148	0.001 187	0.083 947	138.795 453	0.009 695
23	147.088 880	147.090 121	0.001 241	0.085 188	147.080 703	0.009 418
24	155.492 375	155.493 502	0.001 127	0.086 316	155.484 342	0.009 160
25	164.010 873	164.012 044	0.001 170	0.087 486	164.003 125	0.008 919
26	172.641 637	172.642 712	0.001 074	0.088 560	172.634 019	0.008 693
27	181.381 557	181.382 666	0.001 109	0.089 669	181.374 185	0.008 481
28	190.228 211	190.229 239	0.001 027	0.090 697	190.220 957	0.008 282
29	199.178 864	199.179 919	0.001 055	0.091 752	199.171 825	0.008 094
30	208.231 354	208.232 339	0.000 985	0.092 736	208.224 423	0.007 916
31	217.383 254	217.384 262	0.001 008	0.093 744	217.376 514	0.007 747
32	226.632 622	226.633 568	0.000 946	0.094 690	226.625 981	0.007 588
33	235.977 285	235.978 250	0.000 965	0.095 655	235.970 814	0.007 436
34	245.415 488	245.416 399	0.000 911	0.096 566	245.409 107	0.007 292
35	254.945 271	254.946 198	0.000 927	0.097 493	254.939 044	0.007 154
36	264.565 039	264.565 918	0.000 879	0.098 372	264.558 895	0.007 023
37	274.273 015	274.273 908	0.000 892	0.099 264	274.267 010	0.006 897
38	284.067 741	284.068 591	0.000 849	0.100 114	284.061 813	0.006 778
39	293.947 597	293.948 458	0.000 861	0.100 975	293.941 796	0.006 663
40	303.911 244	303.912 066	0.000 822	0.101 797	303.905 514	0.006 553
41	313.957 198	313.958 030	0.000 832	0.102 629	313.951 583	0.006 447
42	324.084 224	324.085 021	0.000 797	0.103 426	324.078 676	0.006 345
43	334.290 957	334.291 762	0.000 806	0.104 232	334.285 515	0.006 248
44	344.576 253	344.577 027	0.000 774	0.105 006	344.570 873	0.006 154
45	354.938 852	354.939 634	0.000 781	0.105 787	354.933 570	0.006 063
46	365.377 692	365.378 444	0.000 752	0.106 539	365.372 468	0.005 976
47	375.891 605	375.892 363	0.000 758	0.107 297	375.886 471	0.005 892
48	386.479 599	386.480 331	0.000 732	0.108 029	386.474 520	0.005 811
49	397.140 589	397.141 327	0.000 737	0.108 767	397.135 595	0.005 732
50	407.873 650	407.874 363	0.000 713	0.109 480	407.868 707	0.005 656

Turschner's energies have the asymptotic expansion:

$$\tilde{E}_n^{(1)} = [\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{2/3}\{1-[324(n+\tfrac{1}{2})^2]^{-1}+O[(n+\tfrac{1}{2})^{-4}]\}$$
$$+c_1(-1)^n[\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{-5/3}\{1+O[(n+\tfrac{1}{2})^{-2}]\}$$

where $c_1 = -2^{-6}3^{5/6}\pi^{4/3}[\Gamma(2/3)]^2 \simeq -0.3293\ldots$. Thus the oscillatory term in Turschner's energies mimics the odd-even alternation in the asymptotic expansion for the exact energies.

For $\nu = 4$, the exact energies $E_n^{(4)}$ have the following asymptotic expansion:

$$E_n^{(4)} = [3\pi^{1/2}(n+\tfrac{1}{2})/R]^{4/3}\{1+(1/9\pi)(n+\tfrac{1}{2})^{-2}+O[(n+\tfrac{1}{2})^{-4}]\}$$

where $R = \Gamma(\tfrac{1}{4})/\Gamma(\tfrac{3}{4}) = 2.958\,675\,119\ldots$. Turschner's energies $\tilde{E}_n^{(4)}$ have the asymptotic expansion

$$\tilde{E}_n^{(4)} = [3\pi^{1/2}(n+\tfrac{1}{2})/R]^{4/3}\{1+(5/162)(n+\tfrac{1}{2})^{-2}+O[(n+\tfrac{1}{2})^{-11/3}]\}.$$

The common leading term in these formulae is just the WKB energy. Since $(9\pi)^{-1} \simeq 0.0354$ and $5/162 \simeq 0.0309$ we see that the second term in the asymptotic expansion of Turschner's energies yields a correction to the WKB energies which is almost correct.

We hope that this article may encourage efforts to place Turschner's approximation on a rigorous footing and to extend it to more general systems.

Acknowledgments

It is a pleasure to thank Michael Aizenman and Barry Simon for useful discussions and H Turschner for sending a preprint of Crowley and Hill's paper. We would also like to thank the Princeton University Computer Center for the use of its facilities.

Appendix 1

A1.1. Exact eigenvalues of $-\mathrm{d}^2/\mathrm{d}x^2+|x|$

Instead of studying this operator on $L^2((-\infty,\infty),\mathrm{d}x)$, it is particularly convenient to consider the corresponding pair of operators defined on $L^2((0,\infty),\mathrm{d}x)$, with either Dirichlet or Neumann boundary conditions at 0. The eigenvalues of the operator with the Neumann boundary condition are the energies of the even parity eigenfunctions of the operator defined on the entire line, while the eigenvalues of the operator with the Dirichlet boundary condition are the energies of the odd parity eigenfunctions. Thus for $x \geqslant 0$, we consider the equation

$$-y''+xy = Ey$$

with boundary conditions $y(\infty)=0$ and $y'(0)=0$ for even solutions (respectively, $y(0)=0$ for odd solutions). Since this reduces to Airy's equation under a translation of the independent variable we have $y(x) = c_1\,\mathrm{Ai}(x-E)+c_2\,\mathrm{Bi}(x-E)$ as the general solution. Imposing the boundary conditions, we find from $y(\infty)=0$ that $c_2=0$ and from $y'(0)=0$ (respectively, $y(0)=0$) that E must satisfy $\mathrm{Ai}'(-E)=0$ (respectively, $\mathrm{Ai}(-E)=0$). Thus one obtains the well known result that the energies of $p^2+|x|$ are the negatives of the real zeros of the Airy function $\mathrm{Ai}(x)$ and its derivative.

In the notation of Abramowitz and Stegun (1964, p 450)

$$a_s = s\text{th zero of } \mathrm{Ai}(x) \qquad \text{and} \qquad a'_s = s\text{th zero of } \mathrm{Ai}'(x),$$

where s is a positive integer. If we now label the energy levels to the full problem arranged in increasing order by E_n, $n = 0, 1, 2, \ldots$, we have $E_{2m} = -a'_{m+1}$ and $E_{2m+1} = -a_{m+1}$ for non-negative integral m. By Abramowitz and Stegun (1964, formulae 10.4.94–95),

$$E_{2m} = g[\tfrac{3}{8}\pi(4m+1)] = g[\tfrac{3}{4}\pi(2m+\tfrac{1}{2})]$$

and

$$E_{2m+1} = f[\tfrac{3}{8}\pi(4m+3)] = f[\tfrac{3}{4}\pi(2m+1+\tfrac{1}{2})],$$

where f and g have the following asymptotic expansions (Abramowitz and Stegun 1964, formula 10.4.105):

$$f(z) = z^{2/3}\{1 + \tfrac{5}{48}z^{-2} + \mathrm{O}(z^{-4})\}$$
$$g(z) = z^{2/3}\{1 - \tfrac{7}{48}z^{-2} + \mathrm{O}(z^{-4})\}.$$

Thus for n even

$$E_n = [\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{2/3}\{1 - \tfrac{7}{48}[\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{-2} + \mathrm{O}[(n+\tfrac{1}{2})^{-4}]\},$$

and for n odd

$$E_n = [\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{2/3}\{1 + \tfrac{5}{48}[\tfrac{3}{4}\pi(n+\tfrac{1}{2})]^{-2} + \mathrm{O}[(n+\tfrac{1}{2})^{-4}]\}.$$

It is noteworthy that the even and odd eigenvalues have different asymptotic expansions in $(n+\tfrac{1}{2})$.

A1.2. Exact eigenvalues of $-d^2/dx^2 + x^4$

Bender *et al* (1977) have found the following asymptotic expansion for the eigenvalues E_n of the anharmonic oscillator $p^2 + x^4$:

$$E_n^{3/4}\{A_0 + A_1 E_n^{-3/2} + \mathrm{O}(E_n^{-3})\} = (n+\tfrac{1}{2})\pi,$$

where $A_0 = R\sqrt{\pi}/3$ and $A_1 = -\sqrt{\pi}/4R$, and $R = \Gamma(\tfrac{1}{4})/\Gamma(\tfrac{3}{4}) = 2.958\,675\ldots$. If we then let $z = (n+\tfrac{1}{2})\pi$ and expand E_n as

$$E_n = Cz^{4/3}\{1 + C_2 z^{-2} + \mathrm{O}(z^{-4})\}$$

it follows that $C = A_0^{-4/3}$ and $C_2 = -\tfrac{4}{3}A_0A_1 = \pi/9$. Hence

$$E_n = [3\sqrt{\pi}(n+\tfrac{1}{2})/R]^{4/3}\{1 + (1/9\pi)(n+\tfrac{1}{2})^{-2} + \mathrm{O}[(n+\tfrac{1}{2})^{-4}]\}.$$

Thus we have obtained an asymptotic expansion for the nth eigenvalue of the anharmonic oscillator $p^2 + x^4$.

A1.3. Large n asymptotics of Turschner eigenvalues

We have seen that

$$\tilde{E}_n^{(\nu)} = (-1)^n[\pi^{1/2}\Gamma(\nu^{-1}+\tfrac{3}{2})/2\Gamma(\nu^{-1}+1)]^{\gamma-1}\Gamma(\gamma)F(-n, \gamma, 1, 2)$$

where $\gamma = 1 + 2\nu/(\nu+2)$. Thus, to determine the large n behaviour of $\tilde{E}_n^{(\nu)}$ it suffices to investigate that of the hypergeometric function $F(-n, \gamma, 1, 2)$. Such an investigation was first carried out by Perron (1916, 1917) who found

$$F(-n,\gamma,1,2)=[(2n)^{-\gamma}/\Gamma(1-\gamma)](1+O(1/n))+(-1)^n[(2n)^{\gamma-1}/\Gamma(\gamma)](1+O(1/n))$$

which gives

$$\tilde{E}_n^{(\nu)}=\left(\frac{\pi^{1/2}\Gamma(\nu^{-1}+\frac{3}{2})n}{\Gamma(\nu^{-1}+1)}\right)^{\gamma-1}(1+O(1/n))+c_1(-1)^n\left(\frac{\pi^{1/2}\Gamma(\nu^{-1}+\frac{3}{2})n}{\Gamma(\nu^{-1}+1)}\right)^{-\gamma}(1+O(1/n))$$

where

$$c_1=\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}\left(\frac{\pi^{1/2}\Gamma(\nu^{-1}+\frac{3}{2})}{2\Gamma(\nu^{-1}+1)}\right)^{2\gamma-1}.$$

In addition, Perron gave series expansions for both $(1+O(1/n))$ factors. However, for the special case which we consider here these series turn out to be particularly simple when expanded in powers of the variable $z=n+\frac{1}{2}$. We will now sketch a direct derivation of this expansion.

It can be shown using integration by parts and the Gauss recursion relations for the hypergeometric function that for all real non-integral γ

$$\Gamma(\gamma)\Gamma(1-\gamma)F(-n,\gamma,1,2)=\mathrm{fp}\int_0^1 t^{\gamma-1}(1-t)^{\gamma}(1-2t)^n\,\mathrm{d}t$$

where 'fp' stands for Hadamard finite part (Lighthill 1958; this is just the Euler integral representation for the hypergeometric function extended to the case where the usual integral diverges at one of the endpoints). Hence

$$\begin{aligned}\Gamma(\gamma)\Gamma(1-\gamma)F(-n,\gamma,1,2)&\\ =\mathrm{fp}\int_0^{1/2} t^{\gamma-1}(1-t)^{-\gamma}(1-2t)^n\,\mathrm{d}t&+\mathrm{fp}\int_{1/2}^1 t^{\gamma-1}(1-t)^{-\gamma}(1-2t)^n\,\mathrm{d}t\\ =\mathrm{fp}\int_0^{1/2} t^{\gamma-1}(1-t)^{-\gamma}(1-2t)^n\,\mathrm{d}t&+(-1)^n\,\mathrm{fp}\int_0^{1/2} u^{-\gamma}(1-u)^{\gamma-1}(1-2u)^n\,\mathrm{d}u\end{aligned}\tag{A1.1}$$

where we have changed variables from t to $u=1-t$ in the second integral.

Now define

$$I(\alpha,\beta)=\mathrm{fp}\int_0^{1/2} t^{\alpha}(1-t)^{\beta}(1-2t)^n\,\mathrm{d}t.$$

Letting $s=-\ln(1-2t)$, we arrive at

$$I(\alpha,\beta)=\tfrac{1}{2}\,\mathrm{fp}\int_0^{\infty}\mathrm{e}^{-s/2}\left(\frac{1-\mathrm{e}^{-s}}{2}\right)^{\alpha}\left(\frac{1+\mathrm{e}^{-s}}{2}\right)^{\beta}\mathrm{e}^{-zs}\,\mathrm{d}s$$

where $z=n+\frac{1}{2}$. When $\beta=-(1+\alpha)$ as is the case for both integrals in (A1.1)

$$\begin{aligned}I(\alpha,-(\alpha+1))&\\ &=\tfrac{1}{2}\,\mathrm{fp}\int_0^{\infty}\mathrm{e}^{-s/2}\left(\frac{1-\mathrm{e}^{-s}}{2}\right)^{\alpha}\left(\frac{1-\mathrm{e}^{-s}}{2}\right)^{-(1+\alpha)}\mathrm{e}^{-zs}\,\mathrm{d}s\\ &=\tfrac{1}{2}\,\mathrm{fp}\int_0^{\infty}(\sinh s/2)^{\alpha}(\cosh s/2)^{-(1+\alpha)}\,\mathrm{e}^{-zs}\,\mathrm{d}s.\end{aligned}$$

When $\alpha > -1$ we may drop the fp since the integral converges in the usual sense. An application of Watson's Lemma then yields the asymptotics of $I(\alpha, -(\alpha+1))$ for large z:

$$I(\alpha, -(\alpha+1)) \sim \Gamma(\alpha+1)(2z)^{-(\alpha+1)}\left\{1 + \sum_{m=1}^{\infty} a_{2m} z^{-2m}\right\}. \tag{A1.2}$$

The salient feature of this asymptotic expansion is the absence of odd-order terms in the series due to the evenness (about $s = 0$) of the function $(s/2)^{-\alpha}(\sinh s/2)^{\alpha}(\cosh s/2)^{-(\alpha+1)}$.

The above asymptotic relation (A1.2) is also valid when $\alpha < -1$ and non-integral. To see this, we expand $(\sinh s/2)$ as a series (in powers of s) with remainder where we include enough terms in the series to ensure that the remainder yields an integral which converges in the usual sense. The integrals over the individual terms of the series may be obtained using $\Gamma(\alpha+1) = \text{fp}\int_0^\infty t^\alpha \, e^{-t}\, dt$ and the integral containing the remainder may be expanded using Watson's Lemma.

Combining the above expansions, we have

$$\begin{aligned}\Gamma(\gamma)\Gamma(1-\gamma)F(-n, \gamma, 1, 2) &= \Gamma(\gamma)[2(n+\tfrac{1}{2})]^{-\gamma}\{1 + O[(n+\tfrac{1}{2})^{-2}]\} \\ &\quad + (-1)^n \Gamma(1-\gamma)[2(n+\tfrac{1}{2})]^{\gamma-1}\{1 + O[(n+\tfrac{1}{2})^{-2}]\}\end{aligned}$$

and we can obtain

$$\begin{aligned}\tilde{E}_n^{(\nu)} \sim &\left[\frac{\pi^{1/2}\Gamma(\nu^{-1}+\frac{3}{2})(n+\frac{1}{2})}{\Gamma(\nu^{-1}+1)}\right]^{\gamma-1}\{1 + \tfrac{1}{24}(2\gamma-3)(2-\gamma)(1-\gamma)(n+\tfrac{1}{2})^{-2} + \ldots\} \\ &+ c_1(-1)^n\left[\frac{\pi^{1/2}\Gamma(\nu^{-1}+\frac{3}{2})(n+\frac{1}{2})}{\Gamma(\nu^{-1}+1)}\right]^{-\gamma}\{1 - \tfrac{1}{24}(2\gamma+1)(\gamma+1)\gamma(n+\tfrac{1}{2})^{-2} + \ldots\}\end{aligned}$$

where c_1 is the constant given by Perron.

A1.4. Large n asymptotics of exact eigenvalues

A higher-order WKB analysis (Bender and Orszag 1978) leads to

$$\begin{aligned}E_n^{(\nu)} = \left(\frac{(\nu+2)\pi^{1/2}(n+\frac{1}{2})}{2R_\nu}\right)^{\gamma-1}&\left\{1 + \frac{1}{3\pi}\frac{\nu(\nu-1)}{(\nu+2)^2}(\tan \pi/\nu)^{-1}(n+\tfrac{1}{2})^{-2}\right. \\ &\left. + o[(n+\tfrac{1}{2})^{-2}]\right\} \qquad \text{for } \nu \geq 2.\end{aligned}$$

where $R_\nu = \Gamma(1/\nu)/\Gamma(1/\nu + \frac{1}{2})$ and $\gamma = 1 + 2\nu/(\nu+2)$. For example, when $\nu = 4$ we obtain

$$E_n^{(4)} = [3\pi^{1/2}(n+\tfrac{1}{2})/R_4]^{4/3}\{1 + (1/9\pi)(n+\tfrac{1}{2})^{-2} + o[(n+\tfrac{1}{2})^{-2}]\}$$

in agreement with our previous calculation.

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